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The initial development of a jet caused by fluid, body and free-surface interaction. Part 1. A uniformly accelerating plate

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The flow field induced by a vertical plate accelerating into a stationary fluid of finite depth with a free surface and a gravitational restoring force is investigated. This is a model problem for some technologically important design issues such as the bow splash of a ship moving at forward speed. Experimentally it is found that a thin jet forms on the plate and rises rapidly upwards. We investigate this jet in the small-time approximation and find an analytical solution for the flow field in which the jet emerges out of a thin region where the horizontal momentum of the main flow is converted by inertial effects into a rising jet.

1. Introduction

There is a long history of experimental and theoretical studies of the interaction between a solid body and a fluid with a free surface. One outstanding, and occasionally dramatic, feature of the fluid/body interaction is that of jet formation. As examples of this, a ship moving at forward speed can carry with it a large bow splash, and the impact of a wave on a tidal barrage (or any form of sea defence) may cause a large jet of water to rise vertically. These jets are of some practical importance as they may affect the stability of a vessel or cause wear and damage to a barrage.

At the present time the effect of these jets is assessed by either scale model experiments or, more commonly, numerical simulations of the flow. For the details of various numerical schemes and an extensive source of references see Greenhow (1993). Any numerical scheme that attempts to follow the evolution of such a jet caused by a body being moved impulsively into the fluid must address the question of what spatial resolution is needed to accurately model the jet. The thickness of a jet forming on an undisturbed free surface is initially zero and subsequently increases. The accuracy and efficiency of the numerical calculation of such a flow may be improved if the order of magnitude of this jet thickness is known *a priori*; a discussion of this issue can be found in Greenhow (1987). It is worth pointing out that for wedge entry problems, which are commonly used to model ship-slamming, the flow is intrinsically self-similar so that the temporal evolution of jet thickness is known *a priori*. However, the resolution of the spatial structure of the jet still poses considerable difficulty. Recent work on this type of flow can be found in Cointe & Armand 1987, Cointe 1989 and Howison, Ockendon & Wilson 1991.

This paper studies the small time evolution of a jet which is formed when a vertical plate is accelerated into a stationary fluid of finite depth with a free surface and gravitational restoring force. It is clear that this model problem is related to the initial motion of a slender ship accelerating from rest, where the flow, in a plane transverse to the motion, is that of a plate moving into a stationary body of fluid with a free surface. This particular jet problem has been studied experimentally by Greenhow & Lin (1983) in the context of wavemakers and ship-slamming, and with particular reference to jet formation by Yong & Chwang (1992). An interesting, although unpublished, theoretical study of this problem was carried out by Peregrine in 1972 who developed a solution that was valid except in a neighbourhood of the free-surface/plate intersection where a singularity developed.

Essentially the same analysis although with a greater variety of plate motions was carried out by Chwang (1983). A method of avoiding the above-mentioned singularity was devised by Roberts (1987) who treated a transient wavemaker problem by an expansion in wavemaker amplitude. A train of very short-wavelength dispersive waves was found near to the moving boundary. This study was extended by Joo, Schultz & Messiter (1990) to include capillary effects and it was shown that the prescription of a contact angle between the wavemaker and fluid could supress the short-wavelength wiggles found by Roberts. One feature of the solutions of Roberts and Joo *et al.* that is of some note is the free-surface slope at the contact point jumps to a finite value in infinitesimally small time. We return to a discussion of this point in §3 of this paper. Most recently this problem has been studied by Frankel (1990) in the context of a slightly compressible fluid. However, in the limit of zero compressibility the singularity that was found by Peregrine reappears at the free surface/plate intersection.

We now readdress the basic problem of a plate accelerating into a fluid at rest with a free surface by using matched asymptotic expansions to construct a uniformly valid small-time solution which holds for arbitrary parameter values. The analysis of a surface-piercing plate impulsively moved into a fluid with constant velocity requires a more sophisticated asymptotic theory. This will be given in Part 2 of this work. We take as a starting point the Euler equations for incompressible inviscid flow and construct an asymptotic solution to these with the time (t) as small parameter. In this particular problem the flow is irrotational and we could work with a velocity potential from the outset. We choose not to do this for two reasons. In physical variables we feel there is rather more insight into the pressure and velocity field which causes the jet. Furthermore, for a slightly compressible fluid which is rapidly accelerated such as water-hammer problems, there is the possibility of a curved shock front being formed. By Crocco's theorem this would generate vorticity and we would then be forced to return to the Euler equations as the irrotationality is lost. The basic outer solution contains a non-uniformity, which manifests itself as a singularity at the free surface/plate intersection, and is caused by the neglect of the fluid inertia near to the plate. On rescaling to a region near to the plate it is necessary to solve the full nonlinear free-surface flow problem. However, the boundary conditions are rather simpler than the original ones and a simple exact solution can be found which represents a thin jet of fluid rising uniformly up the plate. The spatial structure of the flow in this region is found by further considering higher-order terms in this inner region. We conclude by surveying other free-surface flow problems of possible practical importance which can be treated by the methods of this paper.

2. Mathematical analysis

The initial boundary value problem for a semi-infinite strip of inviscid and incompressible fluid which is disturbed from its equilibrium under a gravitational body force by a vertical plate accelerating so as to try to compress the fluid is now

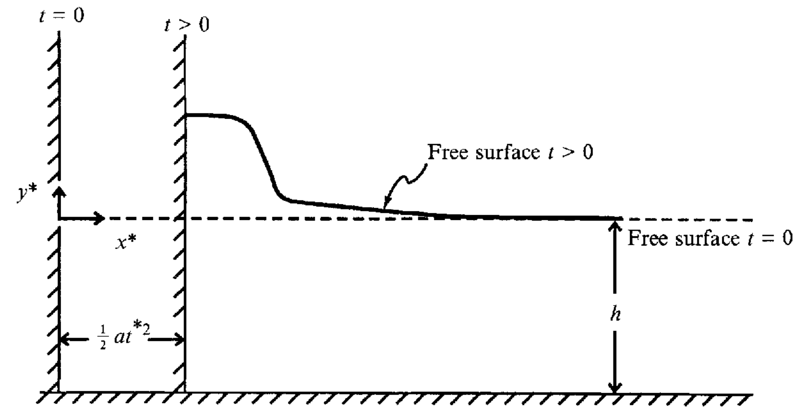

FIGURE 1. Definition sketch of the initial development of the jet on a moving plate.

considered. A definition sketch of the flow is given in figure 1. Initially the fluid is at rest above a rigid bed with depth h. A Cartesian (x^*, y^*) coordinate system with an origin on the undisturbed free surface and x^*-axis in the direction of the plates motion is chosen as shown in figure 1. We find it convenient to work with physical variables so that the Euler equations of fluid motion together with kinematic and zero-pressure free-surface conditions are taken to govern the dynamics of this system.

It is convenient to non-dimensionalize the physical problem using the transformations

$$\boldsymbol{x} = \boldsymbol{x}^*/h, \quad \boldsymbol{q} = \boldsymbol{q}^*/(gh)^{\frac{1}{2}}, \quad t = t^*(h/g)^{\frac{1}{2}}, \quad p = p^*/\rho gh, \tag{2.1}$$

where $\boldsymbol{x}^*$ and $\boldsymbol{q}^*$ are the position and fluid velocity vectors, t^* is time and p^* the pressure. A mathematical statement of the above problem can now be written in the form

$$u_x + v_y = 0, \tag{2.2}$$

$$u_t + uu_x + vu_y = -p_x, \tag{2.3}$$

$$v_t + uv_x + vv_y = -p_y - 1. \tag{2.4}$$

On denoting the free surface by $y = \eta(x, t)$, equations (2.2)–(2.4) are to be solved subject to $\eta(x, 0) = u(x, y, 0) = v(x, y, 0) = 0$ as initial conditions. On the free surface we have

$$v = \eta_t + u\eta_x, \quad p = 0. \tag{2.5}$$

The plate is given a uniform acceleration, $a > 0$, so that its location at time t is given by $x = \sigma t^2$ and we have the further condition on the moving plate,

$$u(\sigma t^2, y, y) = 2\sigma t, \tag{2.6}$$

where the parameter $\sigma = a/2g$ is the ratio of plate acceleration to gravitational acceleration. On the rigid bed we have $v(x, -1, t) = 0$ and as $x \to \infty$ we insist that u, $v \to 0$ and $p \to -y$. The solution domain for this set of equations is unknown at this stage of the analysis but is conveniently described as $D(t) = \{(x, y) : \sigma t^2 \leqslant x \leqslant \infty, -1 \leqslant y \leqslant \eta(x, t)\}$.

A small-time solution to (2.1)–(2.6) may be developed by posing the expansions

$$u = tu_1(x, y) + O(t^2), \quad v = tv_1(x, y) + O(t^2), \tag{2.7a, b}$$

$$p = p_0(x, y) + tp_1(x, y) + O(t^2), \quad \eta = t^2\eta_2(x) + O(t^3) \tag{2.7c, d}$$

as $t \to 0$ with $x = O(1)$, together with a domain decomposition $D(t) = D(0) \cup O(t^2)$

where $D(0)$ is the semi-infinite strip of the undisturbed fluid. At leading order we find the following boundary value problem to solve in $D(0)$:

$$u_{1,x}+v_{1,y}=0, \tag{2.8}$$

$$u_1=-p_{0,x}, \quad v_1=-p_{0,y}-1, \tag{2.9}$$

subject to the conditions

$$u_1(0,y)=2\sigma, \quad v_1(x_1,-1)=0, \quad \eta_2=\tfrac{1}{2}v_1(x,0), \quad p_0(x,0)=0, \tag{2.10}$$

with u_1, $v_1 \to 0$ and $p_0 \to -y$ as $x \to \infty$.

It is clear from (2.8) and (2.9) that p_0 is harmonic in the strip $0 \leqslant x \leqslant \infty$, $-1 \leqslant y \leqslant 0$ and a solution to the above problem may be found by standard separation-of-variable methods in the form

$$p_0=-y-\frac{16\sigma}{\pi^2}\sum_{n=0}^{\infty}\frac{e^{-(n+1/2)\pi x}}{(2n+1)^2}\sin(n+\tfrac{1}{2})\pi y, \tag{2.11}$$

$$\eta_2=\frac{4\sigma}{\pi}\sum_{n=0}^{\infty}\frac{e^{-(n+1/2)\pi x}}{2n+1}. \tag{2.12}$$

The series in (2.12) can be summed exactly to give $\eta_2=(2\sigma/\pi)\ln(\coth\pi x/4)$ which reveals a singularity in the free-surface elevation as $x \to 0$. This singularity is also present in p_0 as $x, y \to 0$ and is compounded by higher-order terms in the expansions (2.7). This non-uniformity in the expansion about the intersection of the plate and free surface reveals (2.7) to be an outer expansion to this problem. To correctly capture the behaviour in the neighbourhood of the plate and free-surface intersection we require an inner region in which $x, y = o(1)$ as $t \to 0$. To motivate the form of the inner expansion we require the local behaviour of the pressure p_0 as $(x^2+y^2)^{\frac{1}{2}} \to 0$. Using the closed-form expression for η_2 above and defining P by $p_0=-y-2\sigma P$ we are led to consider the following boundary problem in the quarter-plane $0 \leqslant r \leqslant \infty$, $-\frac{1}{2}\pi \leqslant \theta \leqslant 0$:

$$\nabla^2 P=0 \quad \text{s.t.} \quad P_\theta=r \quad \text{on} \quad \theta=-\tfrac{1}{2}\pi, \quad P=0 \quad \text{on} \quad \theta=0,$$

and
$$P_\theta=-\frac{2}{\pi}r\ln r+\frac{2}{\pi}\ln\left(\frac{4}{\pi}\right)r+O(r^3) \quad \text{on} \quad \theta=0,$$

where (r,θ) are the standard polar coordinates based at the Cartesian origin. As we are interested in the solution for r small we pose a coordinate expansion in the form

$$P=r\ln r g(\theta)+rh(\theta)+o(r) \quad \text{as} \quad r \to 0, \tag{2.13}$$

and some simple computations show that

$$g(\theta)=-\frac{2}{\pi}\sin\theta \quad \text{and} \quad h(\theta)=\frac{2}{\pi}\left[1+\ln\frac{4}{\pi}\right]\sin\theta-\frac{2}{\pi}\theta\cos\theta.$$

It is useful to record for the purpose of asymptotic matching that will be performed later on that

$$p_0=\frac{4\sigma r}{\pi}\left\{\sin\theta\ln r+\theta\cos\theta-\left(1+\ln\frac{4}{\pi}\right)\sin\theta-\frac{\pi\sin\theta}{4\sigma}\right\}+o(r), \tag{2.14}$$

$$\eta_2=-\frac{2\sigma}{\pi}\ln x+\frac{2\sigma}{\pi}\ln\frac{4}{\pi}+O(x^2) \quad \text{as} \quad x=r \to 0. \tag{2.15}$$

The fluid velocities in the corner region are calculated from $u_1 = 2\sigma P_x$ and $v_1 = 2\sigma P_y$ and indicate that $u_1 = O(1)$ whereas $v_1 = O(\ln r)$ as $r \to 0$. The dominance of vertical velocity over horizontal velocity indicates that near the corner the horizontal momentum imparted to the fluid by the (horizontal) translation of the plate is less important than the presence of a free surface which enables the fluid to escape vertically upwards without having to overcome as much inertia as a horizontal motion.

In order to construct an inner solution to this problem when $x, y = o(1)$ as $t \to 0$, it is useful to examine where the magnitude of terms retained in deriving (2.8) and (2.9) is equal to the terms neglected. The above local analysis shows that as $r = (x^2+y^2)^{\frac{1}{2}} \to 0$; $u = O(t)$, $v = O(t \ln r)$, $p = O(r \ln r)$ and $\eta = O(t^2 \ln r)$. Thus a typical retained term in, say, (2.4) is $v_t = O(\ln r)$ whereas a typical neglected term, which represents fluid inertia, is $vv_y = O((t^2 \ln^2 r)/r)$. These two terms are of equal magnitude when $t^2 \ln r = O(r)$. If this is solved iteratively then it is clear that when $r = O(-t^2 \ln t)$, inertial terms are important and that in this region $u = O(t)$, $v = O(t \ln t)$, $p = O(t^2 \ln^2 t)$ and $\eta = O(t^2 \ln t)$. These estimates motivate the introduction of the following inner variables:

$$X = -\frac{x}{t^2 \ln t}, \quad Y = -\frac{y}{t^2 \ln t}, \quad U = u, \quad V = v, \quad P = p, \tag{2.16}$$

which leads to

$$U_X + V_Y = 0, \tag{2.17a}$$

$$U_t - X\left(\frac{2}{t} + \frac{1}{t \ln t}\right) U_X - Y\left(\frac{2}{t} + \frac{1}{t \ln t}\right) U_Y - \frac{1}{t^2 \ln t} UU_X - \frac{1}{t^2 \ln t} VU_Y = \frac{1}{t^2 \ln t} P_X, \tag{2.17b}$$

$$V_t - X\left(\frac{2}{t} + \frac{1}{t \ln t}\right) V_X - Y\left(\frac{2}{t} + \frac{1}{t \ln t}\right) V_Y - \frac{1}{t^2 \ln t} UV_X - \frac{1}{t^2 \ln t} VV_Y = \frac{1}{t^2 \ln t} P_Y - 1, \tag{2.17c}$$

in $X \geqslant -\sigma/\ln t$, $-\infty \leqslant Y \leqslant \eta(X,t)/-t^2 \ln t$ and subject to free-surface conditions in the form

$$P = 0, \quad V = \eta_t - X\left(\frac{2}{t} + \frac{1}{t \ln t}\right) \eta_X - \frac{1}{t^2 \ln t} U \eta_X. \tag{2.18}$$

The plate condition is $U(-\sigma/\ln t, y, t) = 2\sigma t$, and matching conditions resulting from the expansion of (2.14) and (2.15) when written in terms of the inner variables are to be applied as $(X^2+Y^2)^{\frac{1}{2}} \to \infty$. We proceed with the solution to the inner problem by posing expansions of the form

$$U = t \ln t U_1 + t U_2 + o(t), \quad V = t \ln t V_1 + t V_2 + o(t), \tag{2.19a, b}$$

$$P = t^2 \ln^2 t P_1 + t^2 \ln t P_2 + o(t^2 \ln t), \quad \eta = -t^2 \ln t \eta_1 - t^2 \eta_2 + o(t^2) \tag{2.19c, d}$$

as $t \to 0$ with $X, Y = O(1)$. At leading order we obtain

$$U_{1,X} + V_{1,Y} = 0, \tag{2.20a}$$

$$U_1 - 2XU_{1,X} - 2YU_{1,Y} - U_1 U_{1,X} - V_1 U_{1,Y} = P_{1,X}, \tag{2.20b}$$

$$V_1 - 2XV_{1,X} - 2YV_{1,Y} - U_1 V_{1,X} - V_1 V_{1,Y} = P_{1,Y}, \tag{2.20c}$$

in the domain $0 < X < \infty$, $-\infty < Y < \eta_1$, subject to free-surface conditions on $Y = \eta_1$,

$$P_1 = 0, \quad V_1 = -2\eta_1 + 2X\eta_{1,X} + U_1 \eta_{1,X} \tag{2.21}$$

and a plate condition, $U_1(0, Y) = 0$. Appropriate matching conditions are that

$$\eta_1 \sim \frac{4\sigma}{\pi}, \quad P_1 \sim -\frac{8\sigma Y}{\pi}, \quad U_1 \sim 0, \quad V_1 \sim -\frac{8\sigma}{\pi} \quad \text{as} \quad (X^2+Y^2)^{\frac{1}{2}} \to \infty.$$

The leading-order problem is seen to be the fully nonlinear partial differential equations appropriate to a self-similar unsteady free-surface flow together with boundary conditions which, at this order of approximation, suppress the position of the plate and reflect a uniform high-pressure region deep down in the fluid. The exact solution to this problem is

$$U_1 \equiv 0, \quad V_1 \equiv -\frac{8\sigma}{\pi}, \quad \eta_1 \equiv \frac{4\sigma}{\pi}, \quad P_1 \equiv -\frac{8\sigma}{\pi}\left(Y - \frac{4\sigma}{\pi}\right), \tag{2.22}$$

which represents a block of fluid rising uniformly from the high-pressure zone deep within the fluid. Some support for this form of solution can be found in the photographs of Yong & Chwangs' (1992) paper where a block of fluid with a virtually flat free surface may be seen rising up the plate.

From the point of view of an asymptotic theory we now consider the correction to the leading-order problem and examine it in some detail so as to resolve the spatial structure of the flow near the plate. This resolution is important as it provides a framework for having only two asymptotic regions (provided the correction is bounded) and will give us further specific information, such as the slope of the free surface as it leaves the plate. The next equations in the hierarchy generated by our perturbation process are

$$U_{2,X} + V_{2,Y} = 0, \tag{2.23a}$$

$$U_2 - 2XU_{2,X} - 2YU_{2,Y} + \frac{8\sigma}{\pi} U_{2,Y} = P_{2,X}, \tag{2.23b}$$

$$V_2 - \frac{8\sigma}{\pi} - 2XV_{2,X} - 2YV_{2,Y} + \frac{8\sigma}{\pi} V_{2,Y} = P_{2,Y} - 1, \tag{2.23c}$$

to be solved in the fixed domain $0 < X < \infty, -\infty < Y < 4\sigma/\pi$. The free-surface conditions on $Y = 4\sigma/\pi$ are

$$P_2 = \frac{8\sigma\eta_2}{\pi}, \quad V_2 = -\frac{4\sigma}{\pi} - 2\eta_2 + 2X\eta_{2_X}, \tag{2.24}$$

together with a plate condition $U_2(0, Y) = 2\sigma$ and matching conditions as $(X^2+Y^2)^{\frac{1}{2}} \to \infty$ in the form

$$\eta_2 \sim \frac{2\sigma}{\pi}(\ln X + \lambda), \quad P_2 \sim -\frac{4\sigma}{\pi}\left\{\left(\lambda - 1 - \frac{\pi}{4\sigma}\right)Y + \ln r\, Y + \theta X\right\}, \tag{2.25a, b}$$

$$U_2 \sim -\frac{4\sigma}{\pi}\theta, \quad V_2 \sim -\frac{4\sigma}{\pi}\{\ln r + \lambda\}, \tag{2.25c, d}$$

where $\lambda = \ln(-\ln t) - \ln(4/\pi)$ is regarded as a constant in light of the Van Dyke matching principle as applied to series containing logarithms. Here (r, θ) are the usual polar coordinates with respect to the Cartesian coordinates (X, Y).

An exact solution to the above quarter-plane problem can be found using an integral transform method. The details of this may be found in the appendix to this paper. The solution is found to contain no singularities, so that only two asymptotic regions are

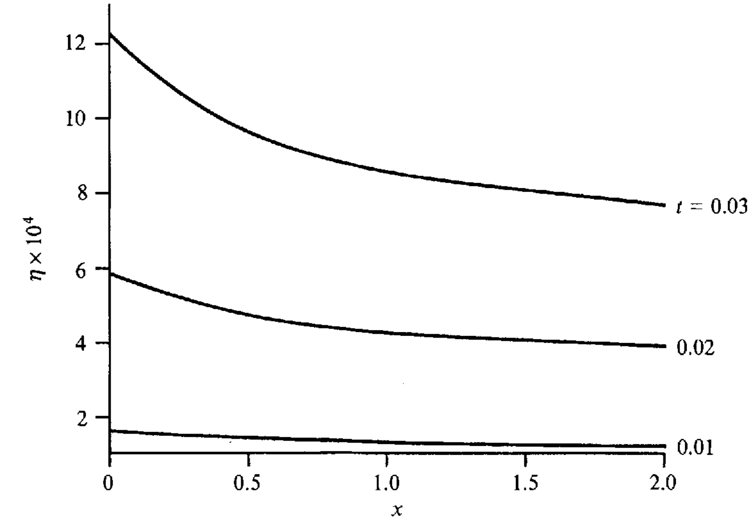

FIGURE 2. The free-surface elevation in the inner region for $\sigma = \frac{1}{4}$ and $t = 0.01, 0.02, 0.03$.

necessary, and confirms our choice of solution of the leading-order problem. The correction to the leading-order free-surface elevation, $\eta_2(X)$ can be written as

$$\eta_2(X) = \frac{2\sigma}{\pi}(\ln X + \lambda) + \tfrac{1}{2}\sigma\pi^{\frac{1}{2}}\frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{(2\sigma/\pi X)^p \,\mathrm{d}p}{p(1+p)\sin(\frac{1}{2}\pi p)\,\Gamma(p+\frac{3}{2})} \tag{2.26}$$

with $0 < c < 2$. The integral appearing in (2.26) is of $O(1/X^2)$ as $X \to \infty$ so that $\eta_2(X)$ matches the outer solution satisfactorily. As $X \to 0$ we can show that

$$\eta_2(X) = \frac{2\sigma}{\pi}\left\{\ln(-\ln t) - \ln\left(\frac{4}{\pi}\right) + \ln\left(\frac{2\sigma}{\pi}\right) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} + \frac{\pi^2 X}{8\sigma}\right\} + o(X). \tag{2.27}$$

Equation (2.27) gives the free surface leaving the plate in a linear manner. A calculation using (2.16), (2.19) and (2.27) shows that the height of the free surface as it leaves the plate is

$$\eta = -\frac{4\sigma}{\pi}t^2 \ln t - \frac{2\sigma t^2}{\pi}\left\{\ln(-\ln t) - \ln\left(\frac{4}{\pi}\right) + \ln\left(\frac{2\sigma}{\pi}\right) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}\right\} + o(t^2) \tag{2.28}$$

and the slope at the free surface, where it intersects the plate, is, at leading order,

$$\eta_x = \pi/(4 \ln t). \tag{2.29}$$

This value is initially zero and become negative for $t > 0$ and may provide a useful check on numerical computations which are performed on this type of free-surface flow problem. A graph of the evolution of the free surface in the inner region is shown in figure 2 for $\sigma = \frac{1}{4}$ and clearly shows the increasing gradient at the free-surface/plate intersection.

3. Conclusions and remarks

The initial development of a jet which is formed by a vertical plate accelerating into a semi-infinite expanse of fluid with a free surface and a gravitational restoring force has been studied asymptotically. The main body of fluid is pushed along horizontally by the plate, with a gradual rise in free-surface elevation as the plate is approached. The outer solution becomes singular at the initial intersection of the plate and free surface owing to the neglect of inertia terms in the asymptotic approximation. In a thin region, of size $O(-t^2 \ln t)$ about this point, vertical motion is dominant as the fluid finds it easier to rise towards the low-pressure free surface rather than overcome the fluids' inertia that it would meet by moving horizontally. These two distinct motions are analysed exactly and blended together using matched asymptotic expansions.

There are further problems that can be addressed by the methods developed in this paper. In the slender-ship approximation the steady three-dimensional flow can be reduced by a perturbation method to unsteady two-dimensional flow of an inclined plate, with a submersion depth less than the fluid depth, which is accelerated into the fluid. In this problem the time variable parameterizes the distance down the longest axis of the ship. A solution to this problem could be used to construct the initial stages of the three-dimensional bow wave of a ship starting from rest. The interaction of a surface-piercing cylinder and a swell may also be of some interest although the analysis in each region of the flow is more complicated.

Finally, it should be noted that the asymptotic analysis presented in this paper gives a uniform spatial approximation to the flow at $t \to 0^+$, with the parameter $\sigma > 0$ fixed. However, as can be seen from (2.19), (2.22), (2.27), the two-term development of η in the inner region is given by

$$\eta \sim -t^2 \log t \frac{4\sigma}{\pi} - t^2 \left\{ \frac{2\sigma}{\pi} \left[\log(-\log t) - \log \frac{4}{\pi} + \log \frac{2\sigma}{\pi} - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} \right] + \frac{\pi^2}{8} X \right\} \qquad (3.1)$$

as $t \to 0$, uniformly for $X \ll 1$. We observe from (3.1) that a non-uniformity develops as $\sigma \to 0$, in particular when $t^2 \log t\sigma \sim t^2$, that is, when $t = O(\mathrm{e}^{-1/\sigma})$. Therefore, the small-time asymptotics we have developed remain uniform in σ provided

$$t \ll \min\{\mathrm{O}(1), \mathrm{O}(\mathrm{e}^{-1/\sigma})\}. \qquad (3.2)$$

For $\sigma \ll 1$ and $t \sim O(\mathrm{e}^{-1/\sigma})$, further consideration is necessary. This observation allows the work of Joo *et al.* (1990) to be placed in context with the present analysis. Joo *et al.* (1990) examine the same problem in the limit $\sigma \to 0$ with $t = O(1)$. However, their asymptotic expansions in σ are non-uniform as $t \to 0^+$: the initial condition that $\eta \to 0$ uniformly in X as $t \to 0^+$ is not satisfied. Indeed their analysis gives a jump $[\eta_X|_{X=0}]_{t=0^-}^{t=0^+} = -\sigma$ so that they obtain $\eta_X|_{X=0} \to -\sigma$ as $t \to 0^+$, and do not satisfy the initial condition on η. It must be concluded that their analysis does not resolve the short-time development of the flow structure, in the neighbourhood of the free-surface/plate intersection point, when $\sigma \ll 1$. In particular, it is to be expected that their analysis fails when $t \sim O(\mathrm{e}^{-1/\sigma})$ as $\sigma \to 0$, when their slope estimate as $t \to 0^+$ $(\eta_X|_{X=0} \to -\sigma)$ becomes $\eta_X|_{X=0} \sim -\sigma = O(-1/\log t)$, which is in accord with (2.29), and an inner temporal region which would allow the initial condition on η to be satisfied is necessary.

Appendix A. Reformulation of a boundary value problem

The boundary value problem in the quarter-plane defined by (2.23), (2.24) and (2.25) is not straightforward to solve. To proceed we define new dependent and independent variables by

$$v = V_2 - \frac{8\sigma}{\pi}, \quad u = U_2, \quad p = P_2 - Y, \quad \eta = \eta_2, \quad y = Y - \frac{4\sigma}{\pi}, \quad x = X. \qquad (\mathrm{A}\ 1)$$

This gives the more symmetric system

$$u_x + v_y = 0, \tag{A 2}$$

$$u - 2xu_x - 2yu_y = p_x, \tag{A 3}$$

$$v - 2xv_x - 2yv_y = p_y \tag{A 4}$$

in the domain $-\infty < y < 0$, $0 < x < \infty$ and subject to the free-surface conditions

$$p(x,0) = \frac{8\sigma\eta}{\pi} - \frac{4\sigma}{\pi}, \quad v(x,0) = -\frac{12\sigma}{\pi} - 2\eta + 2x\eta_x \tag{A 5}$$

and a plate condition $u(0,y) = 2\sigma$. Eliminating the pressure from (A 3) and (A 4) gives a vorticity (ξ) equation in the form

$$\xi + 2x\xi_x + 2y\xi_y = 0, \tag{A 6}$$

where $\xi = u_y - v_x$. Equation (A 6) is readily solved by the method of characteristics, which shows that $d\xi/dx = -\xi/2x$ on the curves $dy/dx = y/x$. As the vorticity is bounded and the outer flow is irrotational the only solution of (A 6) is $\xi \equiv 0$.

This enables us to introduce a velocity potential ϕ such that $u = \phi_x$, $v = \phi_y$. Upon integration of (A 3) and (A 4) we find that $p = p_0 + 3\phi - 2x\phi_x - 2y\phi_y$, where p_0 is a constant pressure which could be found by higher-order matching. As the value of p_0 does not affect the boundedness of our solution or the slope of the free surface we leave it undetermined. Our task now is to solve the quarter-plane problem:

$$\nabla^2\phi = 0, \quad \phi_x(0,y) = 2\sigma, \tag{A 7}$$

with free-surface conditions on $y = 0$ in the form

$$p_0 + 3\phi - 2x\phi_x = \frac{8\sigma}{\pi}\eta - \frac{4\sigma}{\pi}, \quad \phi_y = -\frac{12\sigma}{\pi} - 2\eta + 2x\eta_x \tag{A 8}$$

and a matching condition as $(x^2+y^2)^{\frac{1}{2}} \to \infty$,

$$\phi \sim -\frac{4\sigma}{\pi}\left\{y\ln(x^2+y^2)^{\frac{1}{2}} + x\tan^{-1}\left(\frac{y}{x}\right) + (\lambda+1)\,y\right\} + o((x^2+y^2)^{\frac{1}{2}}) \tag{A 9a}$$

$$\eta \sim \frac{2\sigma}{\pi}(\ln x + \lambda) + o(1). \tag{A 9b}$$

Before using integral transforms to solve this linear boundary value problem, some further manipulations are necessary. We use standard polar coordinates (r,θ) based at the Cartesian origin and redefine the potential and free surface by

$$\bar{\phi} = \phi + \frac{4\sigma}{\pi}\{r\sin\theta\ln r + \theta r\cos\theta + (\lambda+1)\,r\sin\theta\}, \tag{A 10}$$

$$\bar{\eta} = \eta - \frac{2\sigma}{\pi}(\ln r + \lambda), \tag{A 11}$$

so that $\bar{\phi}$ is harmonic with $\bar{\phi}_\theta(r, -\frac{1}{2}\pi) = 0$ and the free-surface conditions can be written as

$$p_0 + 3\bar{\phi} - 2r\bar{\phi}_r = \frac{4\sigma}{\pi}\left\{2\bar{\eta} + \frac{4\sigma}{\pi}(\ln r + \lambda) - 1\right\}, \tag{A 12}$$

$$\frac{1}{r}\bar{\phi}_\theta = -2\bar{\eta} + 2r\bar{\eta}_r. \tag{A 13}$$

The conditions $\bar{\eta} = o(1)$ and $\bar{\phi} = o(r)$ apply as $r \to \infty$. The condition $\bar{\phi} = o(r)$ is still not good enough for a transform technique to be applied. If a coordinate expansion is made for $r \gg 1$ it is found that $\bar{\phi} = (16\sigma^2/3\pi^2)\ln r + B + O(1/r)$, with B a constant related to p_0 and λ, and $\bar{\eta} = O(1/r^2)$. We now make a further redefinition of the potential (to obtain both the potential and free surface vanishing as $r \to \infty$) using

$$\phi^* = \bar{\phi} - \frac{16\sigma^2}{6\pi^2}\ln(1+r^2) - B, \quad \eta^* = \bar{\eta}. \tag{A 14}$$

This choice of ϕ^* is to avoid introducing a singularity for $r \to 0$ but to force $\phi^* = O(1/r)$ as $r \to \infty$ and gives rise to the following boundary value problem:

$$\nabla^2\phi^* = -\frac{16\sigma^2}{6\pi^2}\nabla^2\ln(1+r^2) \tag{A 15}$$

subject to $\phi^*_\theta(r, -\frac{1}{2}\pi) = 0$ and free-surface conditions on $\theta = 0$

$$3\phi^* - 2r\phi^*_r = \frac{8\sigma}{\pi}\eta^* + F^*(r), \tag{A 16}$$

$$\frac{1}{r}\phi^*_\theta = -2\eta^* + 2r\eta^*_r, \tag{A 17}$$

where
$$F^*(r) = \frac{16\sigma^2}{3\pi^2}\left\{3\ln r - \tfrac{3}{2}\ln(1+r^2) + \frac{2r^2}{1+r^2} - 2\right\}.$$

We now construct a solution to this problem with $\phi^* = O(1)$, $\eta^* = O(\ln r)$ as $r \to 0$ and $\phi^* = O(1/r)$, $\eta^* = O(1/r^2)$ as $r \to \infty$ by using Mellin transforms.

The Mellin transforms of ϕ^* and η^* are defined in the usual way as

$$\hat{\phi}(p,\theta) = \int_0^\infty r^{p-1}\phi^*(r,\theta)\,\mathrm{d}r, \quad \hat{\eta}(p) = \int_0^\infty r^{p-1}\eta^*(r)\,\mathrm{d}r.$$

Given $\phi^* = O(1)$ and $\eta^* = O(\ln r)$ as $r \to 0$ and $\phi^* = O(1/r)$ and $\eta^* = O(1/r^2)$ as $r \to \infty$ we expect $\hat{\phi}$ to exist and be analytic in the strip $0 < \mathrm{Re}(p) < 1$ of the complex p-plane; $\hat{\eta}$ will similarly be analytic in $0 < \mathrm{Re}(p) < 2$. Taking transforms of the partial differential equation and using standard results gives

$$\left\{\frac{\partial^2}{\partial\theta^2} + p^2\right\}\hat{\phi} = -\frac{16\sigma^2 p}{6\pi\sin(\frac{1}{2}\pi p)}. \tag{A 18}$$

The solution of this is $\hat{\phi} = A(p)\sin p\theta + B(p)\cos p\theta - 16\sigma^2/(6\pi p\sin(\frac{1}{2}\pi p))$. Application of the boundary condition $\hat{\phi}^*_\theta = 0$ on $\theta = -\frac{1}{2}\pi$ gives the relation

$$A(p)\cos(\tfrac{1}{2}p\pi) + B(p)\sin(\tfrac{1}{2}\pi p) = 0. \tag{A 19}$$

Transforming the free-surface conditions results in

$$\left.\begin{aligned}(3+2p)\,\hat{\phi}(p,0) &= \frac{8\sigma}{\pi}\hat{\eta}(p) - \frac{16\sigma^2(3+2p)}{6\pi p\sin(\frac{1}{2}\pi p)},\\ \hat{\phi}_\theta(p-1,0) &= -2(1+p)\,\hat{\eta}(p).\end{aligned}\right\} \tag{A 20}$$

If $B(p)$ is eliminated between these relations we find that $A(p)$ satisfies the difference equation

$$\frac{A(p)}{A(p-1)} = \frac{4\sigma}{\pi}\left\{\frac{(p-1)\tan(\frac{1}{2}\pi p)}{(p+1)(3+2p)}\right\}. \tag{A 21}$$

The solution to this is readily obtained by standard methods as

$$A(p)=\frac{a(p)(-1)^p\sin(\frac{1}{2}(\pi p)(2\sigma/\pi)^p\Gamma(p)}{\Gamma(p+2)\Gamma(p+\frac{5}{2})},\tag{A 22}$$

where $a(p)$ is a solution of $a(p)/[a(p-1)]=1$ and is as yet undetermined. This solution gives the transform of the free-surface elevation

$$\hat{\eta}(p)=\frac{a(p)(-1)^{p-1}\cos(\frac{1}{2}\pi p)(2\sigma/\pi)^{p-1}}{2p(1+p)\Gamma(p+\frac{3}{2})}.\tag{A 23}$$

To determine $a(p)$ and hence complete the transform solution we firstly examine the behaviour of $\hat{\eta}(p)$ as $|p|\to\infty$ and choose $a(p)$ so as to ensure convergence of the Mellin inversion integral. Using Stirling's approximation to the Γ-function for large $|p|=|\mu+\mathrm{i}\tau|$ we have

$$\hat{\eta}(\mu+\mathrm{i}\tau)=\begin{cases}O(a(\mu+\mathrm{i}\tau)\exp[\mu+(\mu+\mathrm{i}\tau)\ln(2\sigma/\pi)]/\tau^{3+\mu}), & \tau\to+\infty\\ O(a(\mu+\mathrm{i}\tau)\exp[\mu+(\mu+\mathrm{i}\tau)\ln(2\sigma/\pi)-2\pi\tau]/\tau^{3+\mu}), & \tau\to-\infty.\end{cases}\tag{A 24}$$

It is clear from this behaviour that to ensure convergence of the inversion integral we require $\mu>-3$ and

$$a(\mu+\mathrm{i}\tau)=\begin{cases}O(1), & \tau\to\infty\\ O(\mathrm{e}^{2\pi\tau}), & \tau\to-\infty.\end{cases}$$

A function of period 1 which has this property is

$$a(p)=C/[(-1)^p\sin\pi p]\tag{A 25}$$

where C is a constant. The above argument does not exclude the possibility of including within $a(p)$ a further periodic function, $C(p)$, which is bounded above by $O(p^n)$ as $|p|\to\infty$ (with n chosen so as not to interfere with the convergence of the inversion integral). That this function must be constant can be deduced as follows. If $\hat{\eta}(p)$ is to be analytic in $0<\mathrm{Re}(p)<2$, the function $C(p)$ is prevented from having a singularity in this strip. By use of the relation $C(p+1)=C(p)$ we deduce that $C(p)$ is an entire function in the whole p-plane which is bounded above by $O(p^n)$. By Liouville's theorem $C(p)$ must be a polynomial and the only periodic polynomial is a constant. We thus take $C(p)=C$. With this justification for the choice of $a(p)$ we have, using the Mellin inversion formula,

$$\eta^*=\frac{C(2\sigma/\pi)^{-1}}{-4}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{(2\sigma/\pi r)^p\,\mathrm{d}p}{p(1+p)\sin(\frac{1}{2}\pi p)\Gamma(p+\frac{3}{2})}\quad(0<c<2),\tag{A 26}$$

and

$$\phi^*=-\frac{1}{2\pi\mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty}\left[\frac{C\cos p(\theta+\frac{1}{2}\pi)}{p(p+1)\sin(\pi p)\Gamma(p+\frac{5}{2})}+\frac{16\sigma^2}{6\pi}\frac{1}{p\sin\frac{1}{2}p\pi}\right]r^{-p}\,\mathrm{d}p\quad(0<d<1).\tag{A 27}$$

Of particular interest now is the form of free surface that this integral solution represents. The line integral in (A 26) may be turned into a contour integral in $\mathrm{Re}(p)>0$ by noting that the integrand decays on the semicircle $p=R\mathrm{e}^{\mathrm{i}\theta}$, $-\frac{1}{2}\pi<\theta<\frac{1}{2}\pi$ and a simple application of the residue theorem gives

$$\eta^*=\frac{+C}{2\pi}\left(\frac{2\sigma}{\pi}\right)^{-1}\sum_{n=1}^{\infty}\frac{(-1)^n(2\sigma/\pi r)^{2n}}{2n(2n+1)\Gamma(2n+\frac{3}{2})}.\tag{A 28}$$

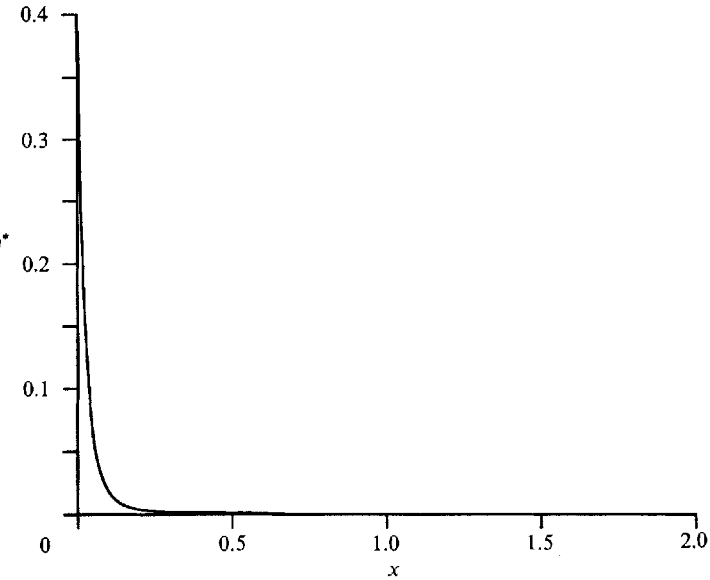

FIGURE 3. The function η^* with $\sigma = \frac{1}{4}$.

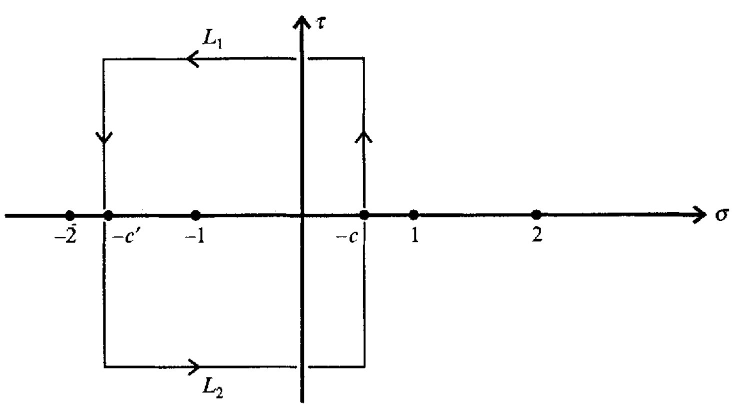

FIGURE 4. The inversion contour for the free-surface elevation in the $p = \sigma + \mathrm{i}\tau$ plane.

The ratio test reveals that this series is convergent for all $r \neq 0$. In fact for large r the series is asymptotic and $\eta^* = O(1/r^2)$ as expected. At $r = 0$ the above series diverges and a rather different approach to the evaluation of η^* is needed for small r. A graph of η^*, computed from (A 28), is given in figure 3 and shows a logarithmic divergence as $r \rightarrow 0$. As the line integral cannot be made into a contour integral by the addition of a semicircle in the left half-p-plane (due to the growth in the gamma function) we consider a rectangular contour as shown in figure 4. The contribution from the line segments L_1 and L_2 can be made arbitrarily small owing to the estimate (A 24) and the use of the ML lemma. Accordingly we obtain

$$\eta^* = \frac{C(2\sigma/\pi)^{-1}}{-4}\left\{+\frac{1}{2\pi \mathrm{i}}\int_{-c'-\mathrm{i}\infty}^{-c'+\mathrm{i}\infty} \frac{(2\sigma/\pi r)^p \,\mathrm{d}p}{p(p+1)\sin\left(\frac{1}{2}\pi p\right)\Gamma\left(p+\frac{3}{2}\right)} + \text{Residues at} \quad p = 0, -1\right\}, \tag{A 29}$$

where $1 < c' < 2$. This integral can be bounded above by

$$\left|\int_{-c'-\mathrm{i}\infty}^{-c'+\mathrm{i}\infty} \frac{(2\sigma/\pi r)^p\,\mathrm{d}p}{p(p+1)\sin(\frac{1}{2}\pi p)\,\Gamma(p+\frac{3}{2})}\right| \leqslant D\left(\frac{\pi r}{2\sigma}\right)^{c'}, \tag{A 30}$$

where D is an $O(1)$ constant. Evaluating the residues at the double and single poles at $p = 0, -1$ leads to the following asymptotic expansion, valid as $r \to 0$:

$$\eta^* = \frac{-C}{4}\left(\frac{2\sigma}{\pi}\right)^{-1}\left\{\frac{2}{\pi\Gamma(\frac{3}{2})}\left[\ln\left(\frac{2\sigma}{\pi r}\right) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})}\right] + \frac{\pi r}{2\sigma\Gamma(\frac{1}{2})}\right\} + o(r). \tag{A 31}$$

Recalling that the physical free-surface elevation was $\eta_2 = (2\sigma/\pi)(\ln r + \lambda) + \eta^*$ we can eliminate the logarithmic term by the choice $C = -(8\sigma^2/\pi)\,\Gamma(\frac{3}{2})$ and the form of free surface close to the plate is seen to be

$$\eta_2 = \frac{2\sigma}{\pi}\left\{\ln(-\ln t) - \ln\left(\frac{4}{\pi}\right) + \ln\left(\frac{2\sigma}{\pi}\right) - 1 - \frac{\Gamma'(\frac{3}{2})}{\Gamma(\frac{3}{2})} + \frac{\pi^2 r}{8\sigma}\right\} + o(r). \tag{A 32}$$

A rather similar analysis may be carried out for the potential ϕ^*. This is found to be regular everywhere. We omit these details as they are similar to the analysis for η^*. We have now established the boundedness of the correction to the leading-order problem over the range $0 \leqslant r < \infty$ and no further asymptotic regions are necessary.

The slope of the free surface as it leaves the plate can now be computed and we find, using $\eta_x = \eta_X X_x = -(1/t^2 \ln t)\{-t^2 \ln t\,\eta_{1,X} - t^2 \eta_{2,X} + \ldots\}$, that the leading-order result is

$$\eta_x = \pi/(4 \ln t) \tag{A 33}$$

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